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A HOLLOW VORTEX WITH AN AXIAL VELOCITY IN A TUBE OF VARIABLE RADIUS†

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The motion of a rotating layer of fluid with a non-zero component of the azimuthal vorticity along the wall of a circular tube of variable radius is considered in the long-wavelength approximation. A system of equations is obtained which is analogous to the equations of turbulent shallow water [1–3]. A rigorous criterion is established which plays a role analogous to the velocity of sound in a gas and which separates solutions showing qualitatively different behaviour. The similarity between the motions of a rotating layer of fluid and the flows of an ideal gas in nozzles is shown.

Equations, analogous to the equations of turbulent shallow water [1] have been previously [4] obtained for tornado-like and hollow vortices in the long-wavelength approximation. These equations were subsequently reduced to a system of integrò-diffferential equations similar to those in [2, 3] (for an infinite hydrodynamic system [2]).

This method is used below to investigate a hollow vortex with an axial velocity in a tube of variable radius. Flows of this kind occur in the nozzles of centrifugal sprayers. The motion of a viscous fluid in a boundary layer in the conical nozzle of a centrifugal sprayer has been considered in [5].

1. FORMULATION OF THE PROBLEM

The steady rotationally-symmetric flow of an ideal incompressible fluid is considered. There is no gravitational force. A cylindrical system of coordinates (r, φ, z) is introduced, with the z-axis directed along the axis of symmetry. The fluid occupies the domain $z \geqslant 0$, $r_1(z) \leqslant r \leqslant r_0(z)$, where $r_0(z)$ is the radius of the tube, which is a specified function of z, and $r_1(z)$ is the radius of the free surface. It is assumed that the flow parameters at $z=0$ are known. The evolution of the flow is investigated as a function of the z coordinate.

Length, velocity and density scales are introduced with the aim of changing to dimensionless quantities. The characteristic scale of the changes along the z-axis is adopted as the unit of length, the magnitude of the rotational component of the velocity when $z=0$ and $r=r_0$ is adopted as the unit of velocity, and the density of the fluid is assumed to be unity. All quantities, even if this is not especially pointed out, are subsequently taken in dimensionless form.

The following notation is introduced (u, v, w) are the velocity components corresponding to (r, φ, z), p is the pressure and δ is the dimensionless value of r_0 when $z=0$. It is assumed that $\delta \ll 1$. In order to change to the long-wavelength approximation, the following extensions of the coordinates and functions [4] are performed

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$$r^2 \to \delta^2\eta,\ z \to z,\ 2ur \to \delta^2 q,\ vr \to \delta A,\ w \to w,\ p \to p$$

In this case, the boundaries $r_1(z)$ and $r_0(z)$ pass into $\eta_1(z)$ and $\eta_0(z)$ and, according to the definition δ, $\eta_0(0)=1$.

As a result, the equations of motion and continuity take the form

$$\begin{aligned} &qA_\eta + wA_z = 0 \\ &(\delta^2/2)(qq_\eta + q^2/(2\eta) + wq_z) - A^2/\eta = -2\eta p_\eta \\ &qw_\eta + ww_z = -p_z,\quad q_\eta + w_z = 0 \end{aligned} \tag{1.1}$$

The corresponding partial derivatives are denoted by the indices of the independent variables.

The boundary conditions

$$p=0,\ q=w\eta_{1z},\ \eta=\eta_1(z) \tag{1.2}$$

$$q=w\eta_{0z},\ \eta=\eta_0(z) \tag{1.3}$$

are imposed on the free surface and the boundary of the tube.

In passing to the long-wavelength approximation, terms in (1.1), which are proportional to δ^2, are omitted. The resulting system is transformed in the same way as in [2–4]. The new independent variables z, ν, $\nu \in [0, 1]$ are introduced, using the relationships $z=z'$, $\eta=R(z', \nu)$, where R satisfies the equation and the boundary conditions

$$wR_{z'} = q;\quad R(z',0)=\eta_0,\ R(z',1)=\eta_1 \tag{1.4}$$

The initial value is $R(0, \nu)$, an arbitrary single-valued continuous function which satisfies the last two equalities in (1.4).

When R is defined in this way, the boundary conditions (1.2) (for q) and (1.3) are automatically satisfied. The unknown boundary $\eta_1(z)$ becomes the known boundary $\nu=1$ and the boundary $\eta_0(z)$ in $\nu=0$. After neglecting terms in δ^2, system (1.1) in the variables z', ν takes the form (the prime on z' is henceforth omitted)

$$\begin{aligned} &wA_z = 0,\ \left(A^2/(2R^2)\right)R_\nu = p_\nu \\ &R_\nu ww_z = -R_\nu p_z + R_z p_\nu,\ q_\nu + R_\nu w_z - R_z w_\nu = 0 \end{aligned}$$

It follows from the first equation that $A=A(\nu)$. Then, using the second equation, we integrate over ν from ν to 1 and eliminate p taking account of the pressure condition (1.2). q is eliminated using (1.4). As a result, after some reduction, we obtain equations in w and R

$$ww_z = \left(A_1^2/(2R_1^2)\right)R_{1z} + \left(\int_\nu^1 (2R)^{-1}A_\nu d\nu\right)_z,\ \left(wR_\nu\right)_z = 0 \tag{1.5}$$

where A_1 and B_1 are the values of A and R when $\nu=1$ (on the free surface).

System (1.5) is solved with the initial data with $z=0$. Let us assume that $w=w_0(\nu)$, $R=1-\varepsilon_0\nu$, when $z=0$, where $\varepsilon_0 = 1-\eta_1(0)$ (account has been taken of the fact that $R(0, 0)=\eta_0(0)=1$ here). The case is later investigated when the circulation A is constant and the vorticity of the flow is due to the non-constancy of the axial velocity. It is initially assumed that the cross-section of the nozzle varies monotonically.

In this case, the following theorem holds.

Theorem. Suppose $A=1$, the function $w_0(\nu)$ is bounded and $w_0(\nu) \geqslant \gamma > 0$ where γ is a constant $\eta_0(z)$ is a monotonic function and

$$\lambda = \frac{1}{2}\varepsilon_0(1-\varepsilon_0)^{-2}\int_0^1 w_0^{-2}d\nu$$

$$\lambda_1 = (1-\varepsilon_0)^2\left[1+(1-\varepsilon_0)\gamma^2\right]^{-2} - \frac{1}{2}\int_0^1 \varepsilon_0 w_0 (w_0^2-\gamma^2)^{-3/2}d\nu$$

It is assumed that $\lambda_1 < 0$. The following cases are distinguished.

1. $\lambda < 1$. Then, if $\eta_0(z) > 0$, a solution exists for all $z \to \infty$ while, if $\eta_0(z) < 0$, a solution exists for $z < z_*$, where z_* is the solution of a certain integral equation and, moreover $w_z \to \infty$ and $R_z \to \infty$ when $z \to z_*$.

2. $\lambda > 1$. Then, if $\eta_0(z) > 0$, a solution exists for $z \leqslant 1$, where l is found from the equation

$$\eta(1) = (1-\varepsilon_0)\left[1+(1-\varepsilon_0)\gamma^2\right]^{-1} + \int_0^1 \varepsilon_0 w_0 (w_0^2-\gamma^2)^{-1/2}d\nu$$

and, when $z = l$, we have $w(\nu) = 0$ for ν specified by the equation $w_0(\nu) = \gamma$. If $\eta_0(z) < 0$, a solution exists for $z \leqslant z_*$ and $w_z \to \infty$ and $R_z \to \infty$ when $z \to z_*$.

Proof. It is assumed that $A = 1$ in (1.5). The system is integrated from 0 to z. An expression for R is found from the second equation by subsequent integration over ν. As a result, we obtain

$$\psi + 1/R_1 = 1/(1-\varepsilon_0), \quad (\psi = w^2 - w_0^2) \tag{1.6}$$

$$R = \eta_0(z) - \int_0^\nu \varepsilon_0 w_0 (w_0^2+\psi)^{-1/2}d\nu$$

Since R_1 is the value of R when $\nu = 1$, it follows from the first equation that $\psi = \psi(z)$, $\psi(0) = 0$ (since $R_1(0) = 1-\varepsilon_0$). After some reduction Eq. (1.6) becomes

$$\eta_0(z) = f(\psi) \tag{1.7}$$

$$f(\psi) = (1-\varepsilon_0)\left[1-(1-\varepsilon_0)\psi\right]^{-1} + \int_0^1 \varepsilon_0 w_0 (w_0^2+\psi)^{-1/2}d\nu$$

The problem has therefore been reduced to finding the implicit dependence $\psi(z)$ which is given by (1.7). The function $f(\psi)$ is considered in the interval $-\gamma_2 \leqslant \psi < 1/(1-\varepsilon_0)$. It is obvious that $f(\psi) > 0$ in this interval. It can be shown by differentiation that $f'(\psi) > 0$. It follows from this that $f(\psi)$ is a monotonically increasing function. $f'(-\gamma^2) = \lambda_1 < 0$, $f'(0) = (1-\varepsilon_0)^2(1-\lambda)$ follows from the expression for $f'(\psi)$.

1. Let $\lambda < 1$. In this case, by virtue of the monotonic increase in $f'(\psi)$ and the inequality $f'(-\gamma^2) < 0$, a graph of the function $f(\psi)$ has the form which is shown qualitatively in Fig. 1. It follows from this graph that, if $\eta_0(z) > 0$ (the nozzle is expanding), then $f(\psi)$, ψ and, consequently, w increase as z increases and a solution exists for all z. If $\eta_0(z) < 0$ (the nozzle becomes narrower), then $f(\psi)$, ψ and, consequently, w decrease as z increases and a solution only exists for $z \leqslant z_*$, $\eta_0 \geqslant \eta_*$, $\psi \geqslant \psi_*$, where ψ_* is found from the equation $f'(\psi_*) = 0$, and z_* and η_* are found from (1.7) after putting $\psi = \psi_*$ in it. On differentiating $f(\psi)$ as a complex function of z and taking account of (1.7), we obtain

$$\psi_z = \eta_{0z}/f_\psi \tag{1.8}$$

By virtue of the fact that $\eta_0 < 0$, $f' \geqslant 0$, it follows from this that $\psi_z \to -\infty$ when $z \to z_*$. Then, according to the definition of ψ and (1.6), we have $w_z \to -\infty$ and $R_z \to -\infty$ when $z \to z_*$.

2. Let $\lambda > 1$. In this case $f'(0) < 0$ and the function $f(\psi)$ has a minimum to the right of the origin of coordinates. It follows from this that, if $\eta_0(z) > 0$ (the nozzle is expanding), then $f(\psi)$ increases, while ψ and w decrease and a solution then exists until ψ reaches a value of $-\gamma^2$. Hence, a solution exists for $z \leqslant l$, where l is calculated from (1.7) with $\psi = -\gamma^2$. It follows from the definition of w that, when $z = l$, we have $w = 0$ for ν specified by the equation $w_0(\nu) = \gamma$.

If $\eta_0(z) < 0$ (the nozzle becomes narrower), then $f(\psi)$ decreases while ψ and w increase and a solution exists for $z \leqslant z_*$, $\eta \geqslant \eta_*$, $\psi \leqslant \psi_*$, where ψ_* is found from the equation $f'(\psi_*) = 0$, and z_* and η_* are found from (1.7) after putting $\psi = \psi_*$ in it. It follows from (1.8), the definition of ψ and (1.6) that $\psi_z \to \infty$, $w_z \to \infty$, $R_z \to \infty$ when $z \to z_*$.

For mathematical generality, we will now consider the case when $\lambda_1 > 0$. Such a flow is hardly physically real since the function $(w_0^2 - \gamma^2)^{-3/2}$ must be integrable in $[0, 1]$ subject to this condition.

Assertion. Let the conditions of the theorem, with the exception of the inequality $\lambda_1 < 0$, be satisfied. Let us put $\lambda_1 \geqslant 0$. Then, if $\eta_0(z) > 0$, a solution exists for all $z \to \infty$ and w increases as z increases. If $\eta_0(z) < 0$, then a solution exists for $z \leqslant l$, where l is found from Eq. (1.7) with $\psi = -\gamma^2$. Here, w decreases as z increases and, when $z = l$, $w = 0$ for ν which are specified by the equation $w_0(\nu) = \gamma$..

The proof follows from the fact that the function $f(\psi)$ is monotonic in the interval $[-\gamma^2, 1/(1-\varepsilon_0)]$ when $\lambda \geqslant 0$.

We note an interesting result of the theorem. In the physically important case when $\lambda_1 < 0$, the behaviour of the flow is qualitatively different in a contracting and in an expanding nozzle. In a contracting nozzle, a solution always exists just for finite $z \leqslant z_*$ and the solution ceases to exist on account of the derivatives becoming infinite. In an expanding nozzle, when $\lambda < 1$, a solution exists for any $z \to l$ while, when $\lambda > 1$, a solution exists up to $z \leqslant l$, where l is found from Eq. (1.7) with $\psi = -\gamma^2$ and a solution ceases to exist due to the axial velocity vanishing for ν specified by the equation $w_0(\nu) = \gamma$.

The results which have been obtained using the established properties of the function $f(\psi)$ and relationships (1.7) can be generalized to the case of a non-monotonic change in the cross-section of the tube. For instance, when $\lambda > 1$, a solution exists for those z for which the inequalities are satisfied: if $\lambda_1 < 0$ (Fig. 1), then $\eta_0(z) \geqslant f(\psi_*)$, if $\lambda_1 \geqslant 0$, then $\eta_0(z) \geqslant f(-\gamma^2)$. When $\lambda > 1$, a solution exists until when $f(-\gamma^2) \leqslant \eta_0(z) \leqslant f(\psi_*)$.

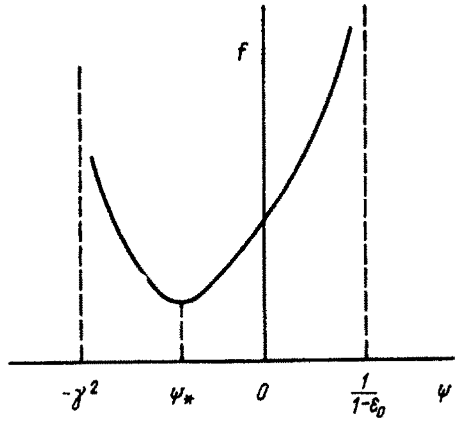

Fig. 1.

2. DISCUSSION OF THE RESULTS. ANALOGY WITH THE FLOWS OF AN IDEAL GAS

It has been shown [4] that, in the long-wavelength approximation, the equations of motion of a fluid in a hollow vortex with a constant circulation are analogous to the equation of turbulent shallow water [2, 3] and, consequently, possess the same hyperbolic properties [3]. Using this, let us find the condition under which the velocity of a characteristic of the discrete spectrum is equal to zero. On putting $A=1$ in (1.5) and allowing for the fact that $R=\eta_0(z)$ when $\nu=0$, we obtain the equations

$$2R_1^2 ww_z - \int_0^1 R_{\nu z} d\nu = \eta_{0z}, \quad R_\nu w_z + wR_{\nu z} = 0.$$

From this, just as in the well-known approach in [3], we obtain the required condition

$$-\int_0^1 \frac{R_\nu}{2w^2} d\nu = R_1^2 \tag{2.1}$$

Substitution of the expressions for w in terms of ψ, R_1 and R_ν, calculated from (1.6) into (2.1) shows that relationship (2.1) is equivalent to the condition $f'(\psi)=0$. Hence, in a contracting nozzle, a solution ceases to exist at the point where the velocity of the characteristic in the fixed system in which it is measured is equal to zero.

Note the analogy between the flows of an ideal gas and the flow of a layer of a rotating fluid in nozzles. The motion of the fluid when $\lambda<1$ (Fig. 1) is analogous to the supersonic flow of a gas while, when $\lambda>1$ (the minimum of the function $f(\psi)$ lies to the right of the origin of coordinates) it is analogous to the subsonic flow of a gas. In the first case, there is an acceleration of the flow in an expanding nozzle and a solution exists up to $z\to\infty$. In a converging nozzle there is a retardation of the flow and a solution ceases to exist when the velocity of the gas becomes equal to the local velocity of sound and, in the case of a layer of fluid, the velocity of the characteristic in the fixed frame of reference becomes equal to zero. These conditions are mathematically equivalent since the velocity of sound in a gas is also the velocity of the characteristic. In the second case (subsonic flow), the flow slows down in an expanding nozzle and a solution exists until its velocity vanishes at some point. In a converging nozzle, there is an acceleration of the flow and a solution exists until the velocity of the gas becomes equal to the velocity of sound or the velocity of the characteristic vanishes.

In this connection, it may be postulated that a phenomenon must occur which is similar to the formation of shock waves in gas dynamics during the flow of a rotating layer of a fluid when the velocity of the layer becomes equal to the local velocity of the characteristic. This may be a "jump" of the vortex (which is analogous to a hydraulic jump in shallow water flows). If retardation of the flow occurs and the vertical velocity vanishes somewhere within the layer, "destruction" of the vortex can occur at this site.

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